Gravitational 2-body problem: III

- constructing orbits from the relative position vector
- unbound orbits and "scattering kinematics"
- "gravity assist" for spacecraft

Let's describe the motion of our two bodies in the "center-of-mass frame", so $\dot{\vec{R}}=0$, and place the position of the center-of-mass at the origin, $\vec{R}=0$. In this coordinate system

$$\vec{r}_1 = \left(\frac{M_2}{M_1+M_2}\right)\vec{r}, \quad \vec{r}_2 = -\left(\frac{M_1}{M_1+M_2}\right)\vec{r}.$$

The two bodies have positions parallel to the relative position vector $\vec{r}$, with mass-ratio dependent scale factors. At any time, the more massive of the bodies will be closer to the origin (center of mass) than the lighter body.

Let's construct the trajectories of a binary star system with equal mass stars, so

$$\vec{r}_1 = \frac{1}{2}\vec{r} = -\vec{r}_2 .$$

Also, suppose $\epsilon = \sqrt{3}/2$ so

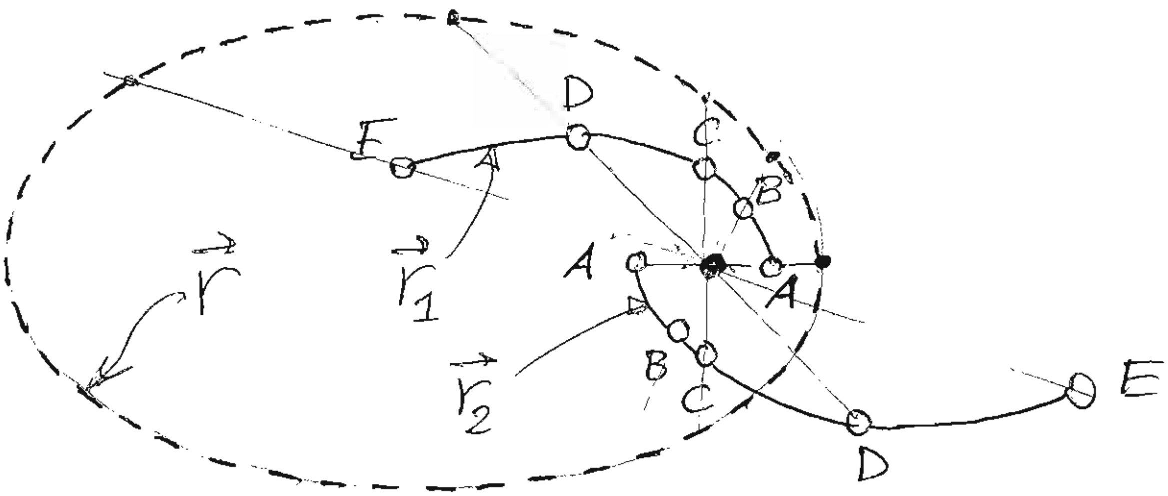
$$a : b = \frac{1}{\sqrt{1-\epsilon^2}} : 1 = 2 : 1$$

We start by drawing the orbit

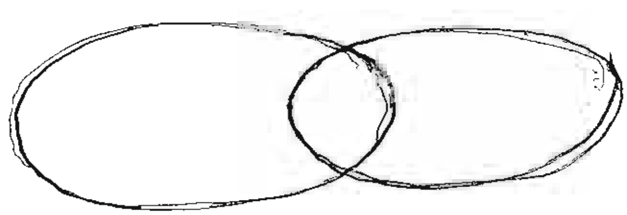
of the relative position $\vec{r}$, an ellipse with aspect ratio 2.

The origin ($\vec{R}=0$) divides the ~~8~~ major axis into the ratio

$$\frac{r_{max}}{r_{min}} = \frac{1+\epsilon}{1-\epsilon} \cong 14$$



A time-lapse photo of the binary star system would look like this:



We now turn to orbits with $\epsilon > 1$, for which the energy $E = (\epsilon^2 - 1)\frac{A}{2r_0}$ is positive, thereby allowing the bodies to "escape to infinity".

Redoing the calculation for the orbit shape, $\quad x_0 = \left(\frac{\epsilon}{\epsilon^2 - 1}\right) r_0$

$$-(\epsilon^2 - 1)(x - x_0)^2 + y^2 = -\frac{r_0^2}{\epsilon^2 - 1},$$

with new definitions

$$a = \frac{r_0}{\epsilon^2 - 1}, \quad b = \frac{r_0}{\sqrt{\epsilon^2 - 1}},$$

we obtain

$$\frac{(x - x_0)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

which we recognize as the equation for a <u>hyperbola</u>.

Only one branch of the hyperbola corresponds to the actual orbit. To see which one, return to the orbit equation before it was squared:

$$r(\theta) = \frac{r_0}{1+\epsilon\cos\theta}$$

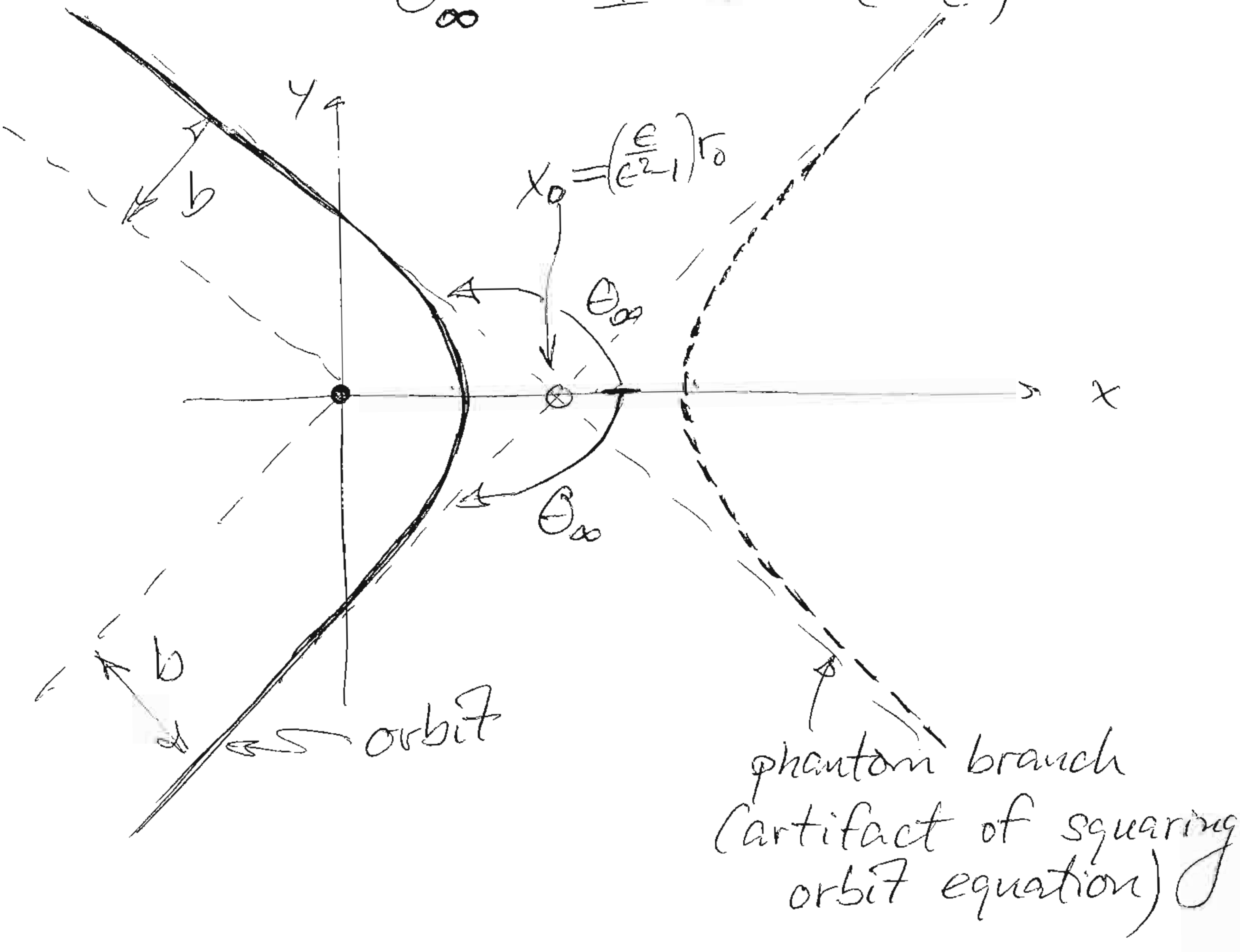
Since $A = GM_1M_2 > 0$, $r_0 = \frac{L_z^2}{A\mu}$ is <u>positive</u>, so the valid range of angles is given by

$$\cos\theta > -1/\epsilon$$

The angles of the asymptotes

that define the approximately "free" motion far from the origin are

$$\theta_\infty = \pm \cos^{-1}(-1/\epsilon)$$



Notice that the orbit asymptotes do <u>not</u> pass through the origin, but the point $(x_0, 0)$. This is necessary if our unbound orbit has $L_z \neq 0$. The parallel shift of asymptotes, relative to the origin, is called the "impact parameter" b (not to be confused with the semi-minor axis of an elliptic orbit).

We can relate the impact parameter to the other orbital parameters using asymptotic expressions for the conserved L_z and E:

$$L_z = b\mu v_\infty = (\vec{r} \times \vec{p})_z$$

$$E = \frac{1}{2}\mu v_\infty^2$$

v_∞ is the speed $|\dot{\vec{r}}|$ of the relative separation in the limit of infinite r. Eliminating v_∞ between these equations and using earlier formulas for r_0 and E,

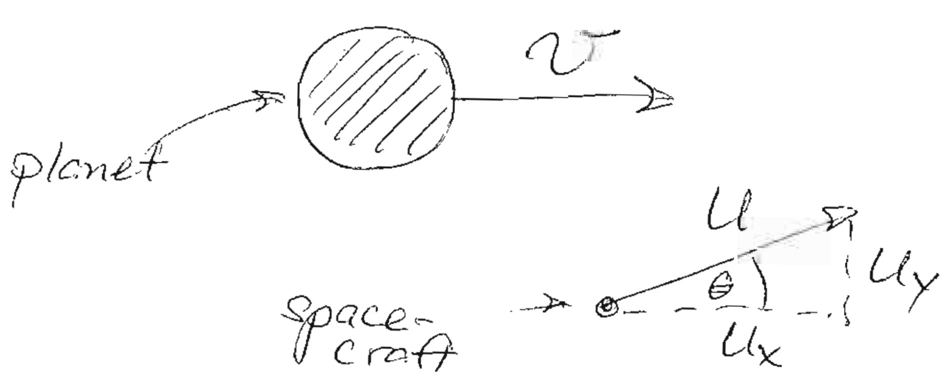
$$\frac{L_z^2}{E} = 2\mu b^2$$

$$\Rightarrow b^2 = \frac{L_z^2}{2\mu E} = \frac{A\mu r_0}{\frac{A\mu}{r_0}(\epsilon^2-1)} = \frac{r_0^2}{\epsilon^2-1}$$

Thus the "impact parameter b" is identical to the "b we defined in the equation for the hyperbola".

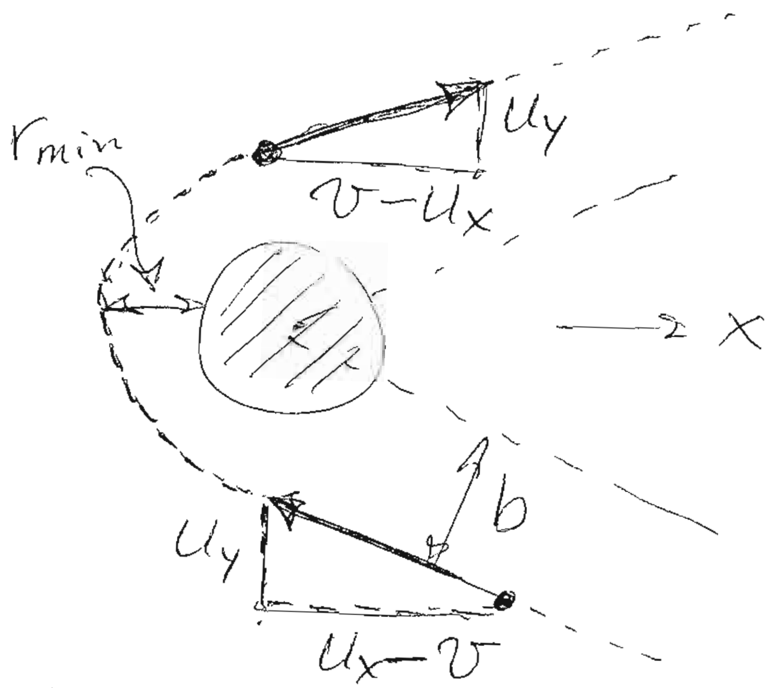
The term "scattering" is usually used to describe the situation we have in the case of unbound orbits, where one state of uniform (unaccelerated) motion is transformed to a different state of uniform motion. A nice example of "scattering" in the gravitational context is the process called "gravity assist", where the energy of a spacecraft is boosted by a well orchestrated sequence of scatterings from planets.

Consider a spacecraft in a near collision course with a planet:



In this scenario $v > u_x$, i.e. the planet is catching-up with the spacecraft. Let's study this in the center-of-mass frame, which, because of the planet's much greater mass, is the frame where the planet is practically at rest:

CoM frame



The impact parameter of the orbit, b, was chosen* so that the x-axis of the hyperbolic trajectory coincides with the planet's direction of motion

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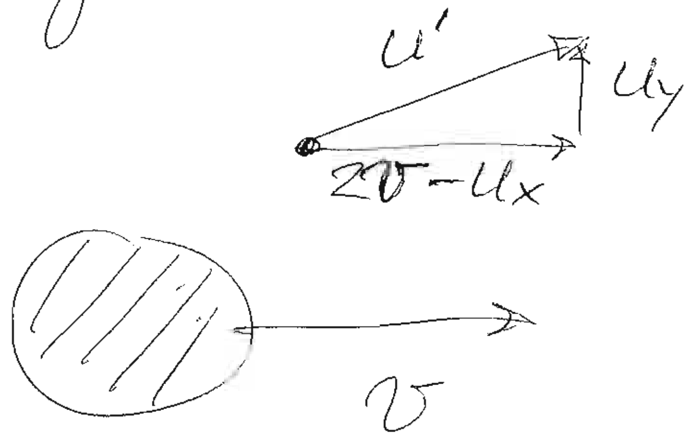
* by mission-control

Going back to the Solar-system frame, we find the following for the spacecraft velocity after scattering:



Comparing final and initial spacecraft speeds:

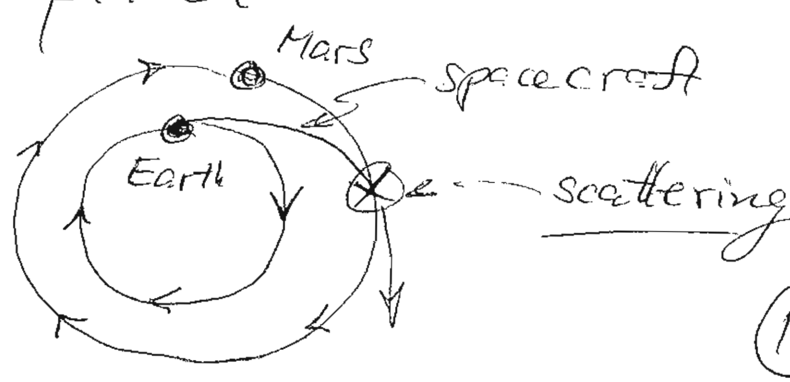
$$\left(\frac{u'}{u}\right)^2 = \frac{(2v-u_x)^2+u_y^2}{u^2}$$

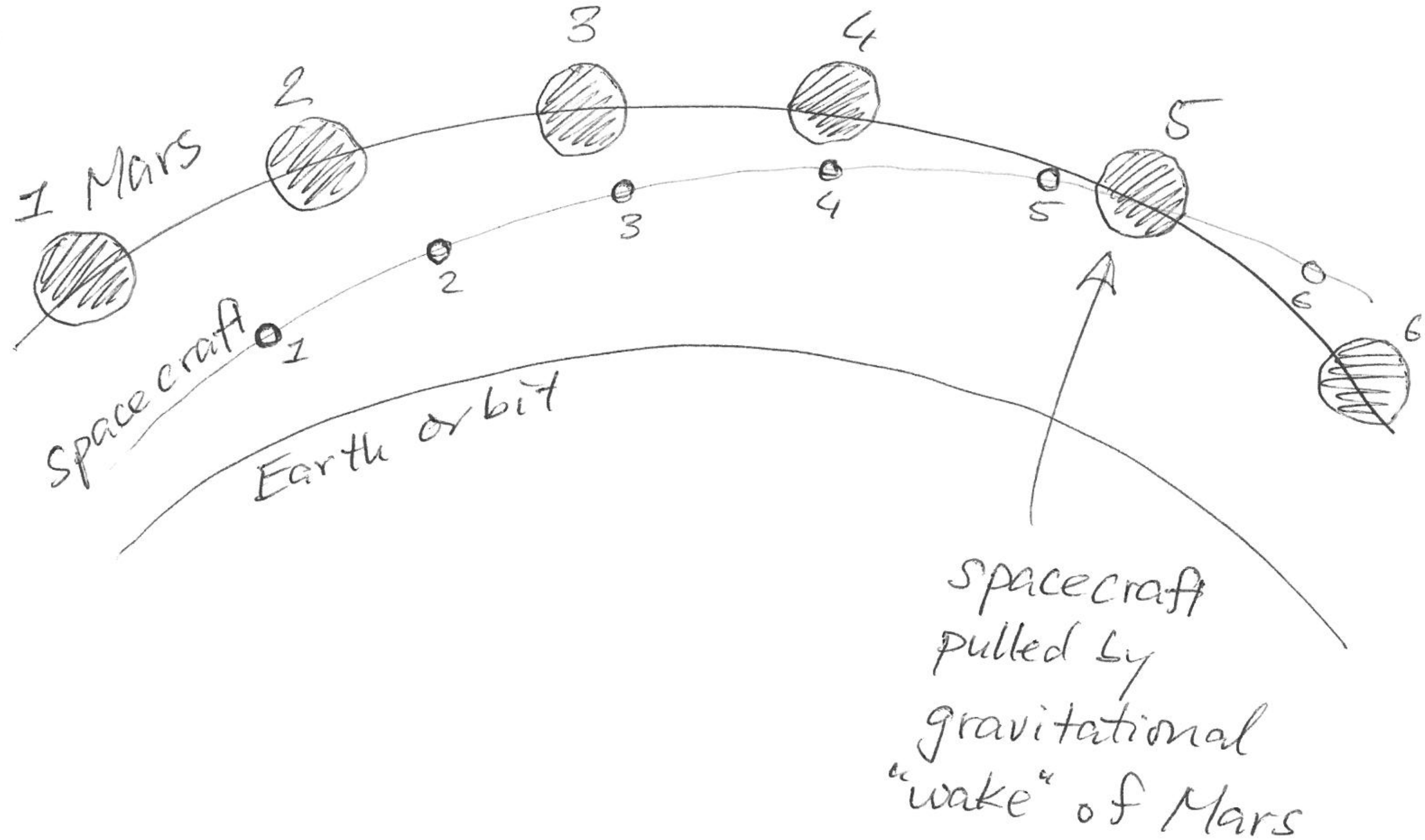
$$= \left(\frac{2v}{u}\right)^2 - 4\frac{v}{u}\underbrace{\frac{u_x}{u}}_{\cos\theta} + 1$$

$$= \left(\frac{2v}{u}-1\right)^2 + 4\frac{v}{u}(1-\cos\theta)$$

We see that $u' > u$ as long as $v > u$, the speed boost being greater for small incident angle θ. Question: How is it that energy is still preserved overall?

A succession of "gravity assist" scatterings was used to propel the voyager spacecraft to the outer planets and beyond. In "climbing out" of the sun's gravitational "well" the spacecraft loses speed, which is then replenished when it gets pulled along by the gravitational wake of a faster moving planet





orbit analysis on two length/time scales:

large: elliptical orbits around sun

small: hyperbolic scattering orbit of spacecraft relative Mars